University of North Georgia
Department of Mathematics

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Course: College Algebra Math 1111

Text Book: For this course we use the free e – book by Stitz and Zeager with link:
http://www.stitz-zeager.com/szca07042013.pdf

Tutorials and Practice Exercises

- http://www.wtamu.edu/academic/anns/mps/math/mathlab/col_algebra/index.htm
- http://www.mathwarehouse.com/algebra/
- http://www.ixl.com/math/algebra-2
- http://www.ixl.com/math/precalculus
- http://www.ltcconline.net/greenl/java/index.html

For more free supportive educational resources consult the syllabus

Chapter 4
Rational Functions (page 301)

4.1 Rational Functions

Objectives: By the end of this section students should be able to**:**

- Define a rational function and give example
- Identify vertical, horizontal and oblique asymptotes
- Find vertical Horizontal and oblique asymptotes
- Sketch graphs of rational functions
- Solve application problems

Rational Function and Asymptotes

Definition: A function f defined by $f(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$, $p(x)$ and $q(x)$ are polynomials is called a **rational function.**

Definition: (Domain**)**
The Domain of a rational function f is set of all **inputs** x for which $q(x) \neq 0$.
That is **Domain of** $f = \{x : q(x) \neq 0\}$
Example 4.1.1: Page 301
Example: Find the domain of: a) $f(x) = \frac{x+1}{x^2-4}$ b) $f(x) = \frac{x}{x^2-2}$ c) $f(x) = 1/x$

A simple Rational Function

Example 1: Graph the function $f(x) = \frac{1}{x}$
The function f is not defined at $x = 0$, so, the **domain** of $f = \{x : x \neq 0\}$

The following two tables show that when **x is close to zero**, $|\ f(x)\ |$ gets large

Table 1	
x	$f(x)$
-0.1	-10
-0.01	-100
-0.001	-1000
-0.00001	-100000
Approaches 0^-	**Approaches to $-\infty$**

Table 2	
x	$f(x)$
0.1	10
0.01	100
0.001	1000
0.00001	100000
Approaches 0^+	**Approaches to ∞**

We describe this behavior as follows
The first Table $f(x) \to -\infty$ as $x \to 0^-$; The second Table $f(x) \to \infty$ as $x \to 0^+$

The next two tables shows how $f(x)$ changes as $|\, x \,|$ becomes large

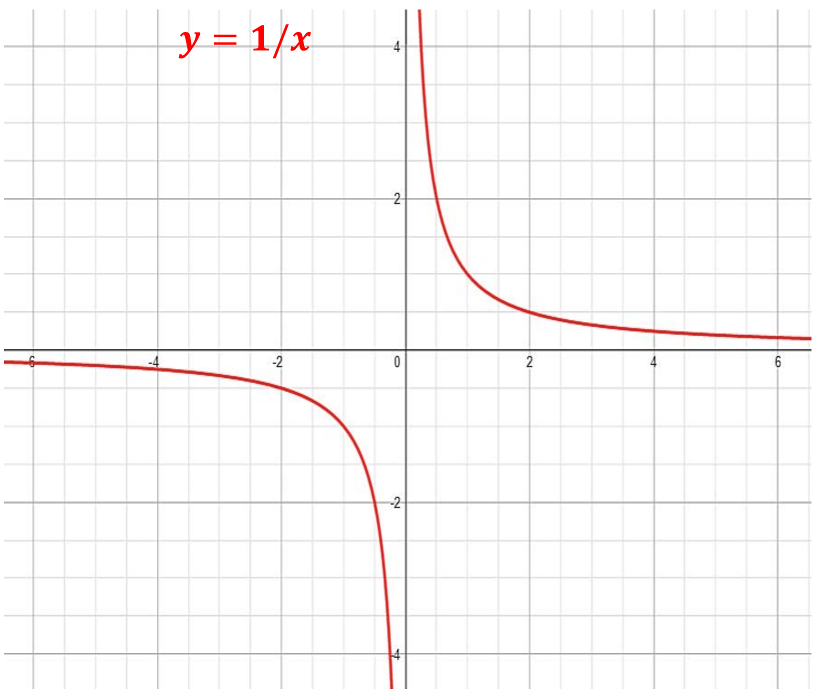
Table 1	
x	$f(x)$
-10	-0.1
-100	-0.01
-100000	-0.00001
Approaches $-\infty$	**Approaches to 0**

Table 2	
x	$f(x)$
10	0.1
100	0.01
100000	0.00001
Approaches ∞	**Approaches to 0**

The Tables shows that as $|\, x \,|$ becomes large, the value of $f(x)$ gets closer and closer to **Zero**
That is:

$$f(x) \to 0 \text{ as } x \to -\infty \text{ and } f(x) \to 0 \text{ as } x \to -\infty$$

Using the information in these Tables and plotting few additional points, we obtain the graph of $y = 1/x$ as shown below



Example 2: Find the domain and sketch the graph using transformation properties on $f(x) = \frac{1}{x}$

a) $f(x) = \frac{2}{x-4}$ **Answer** D = $(-\infty,4) \cup (4,\infty)$

b) $y = \frac{x+1}{x-2}$,

c) $y = \frac{1}{x+5}$

Example 3: In each of the following, which values of x may not be included in the **domain**? That is, which values are the **singularities** of the function? What is the domain of the function?

a) $y = \frac{1}{2x + 1}$

b) $y = \frac{1}{x^2 - 16}$

c) $f(x) = \frac{1}{x^2 + x - 6}$

Asymptotes of Rational Functions

We consider three types of asymptotes: Vertical, Horizontal, and Oblique or Slant Asymptotes

Arrow Notations:

Symbol	Meaning
$x \to a^-$	x **approaches** a from the **left**
$x \to a^+$	x **approaches** a from the **right**
$x \to -\infty$	x goes to **negative infinity**; x **decreases** without bound
$x \to \infty$	x goes to **infinity**; x **increases** without bound

Definition of Vertical and Horizontal Asymptotes

1. **Vertical Asymptote (VA)** is a vertical line; that is a line perpendicular to the x – axis.
   The line $x = a$ is a **Vertical Asymptote** of the function $y = f(x)$ if y **approaches** to $\pm\infty$ as x approaches a from the **right** or from the **left**

Using Arrow Notations

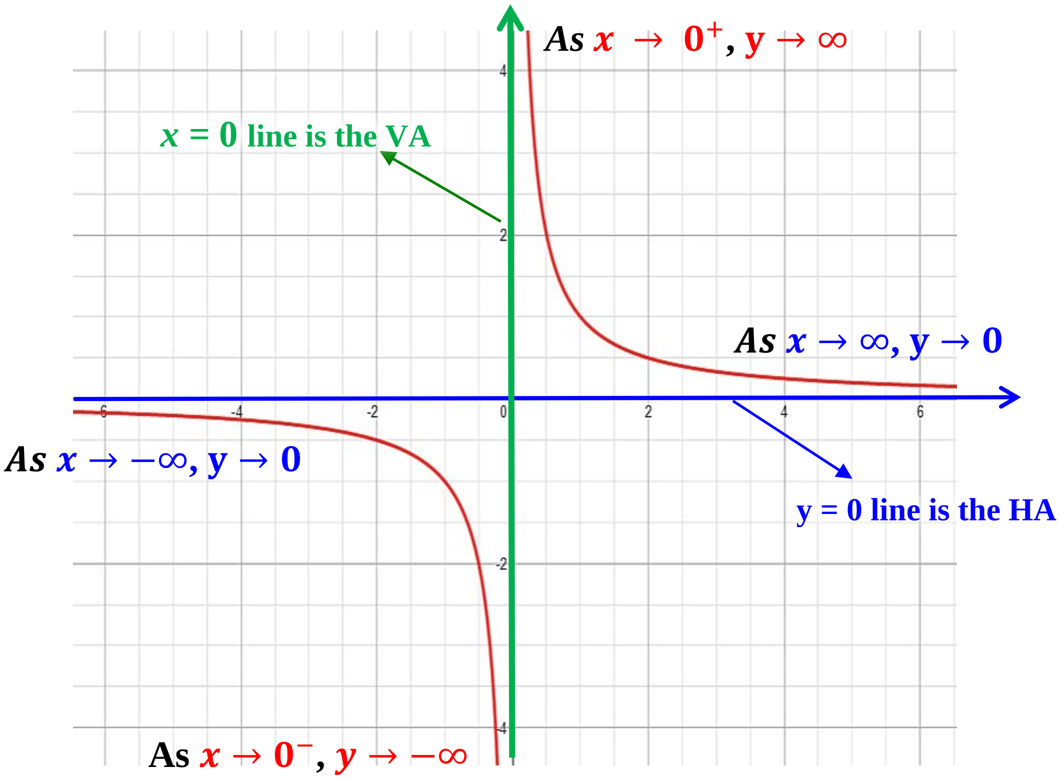
The line $x = a$ is a **Vertical Asymptote** of the graph of a function $y = f(x)$ if as $x \to a^-$ or as $x \to a^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$

2. **Horizontal Asymptote (HA)** is a horizontal line; that is a line parallel to the y – axis
   The line $y = b$ is a **Horizontal Asymptote** of the function $y = f(x)$ if y **approaches** b as x approaches $\pm\infty$

Using Arrow Notations

The line $y = b$ is a **Horizontal Asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to b$

Example 4: **Vertical** and **Horizontal** Asymptotes for the graph of $y = \frac{1}{x}$, see fig below



Question 1: Where do we always find a vertical asymptote of a graph? **At a singularity**

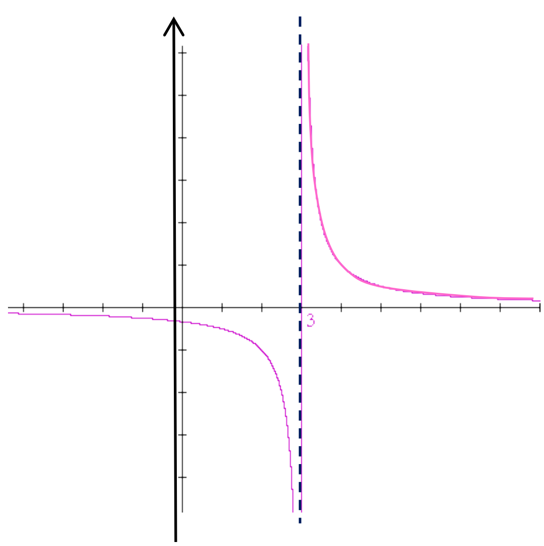
Question 2: What does the equation of a **vertical line** look like? ***x* = A number**

Question 3: What does the equation of a **horizontal line** look like? ***y* = A number**

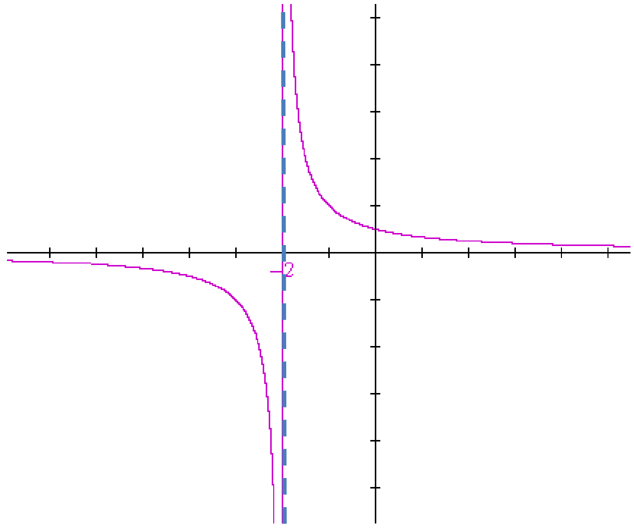
Example 5: Each of the following graphs is a **translation** of the graph of $y = \frac{1}{x}$.

a) $y = \frac{1}{x-3}$

What is the HA and VA?

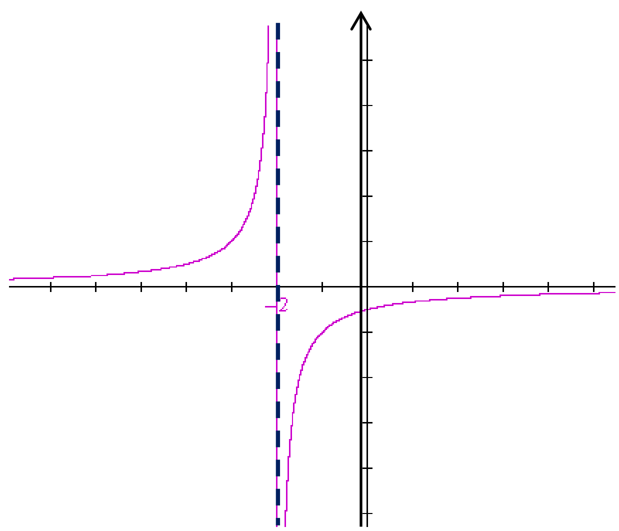


b) $y = \frac{1}{x+2}$



What is the HA and VA?

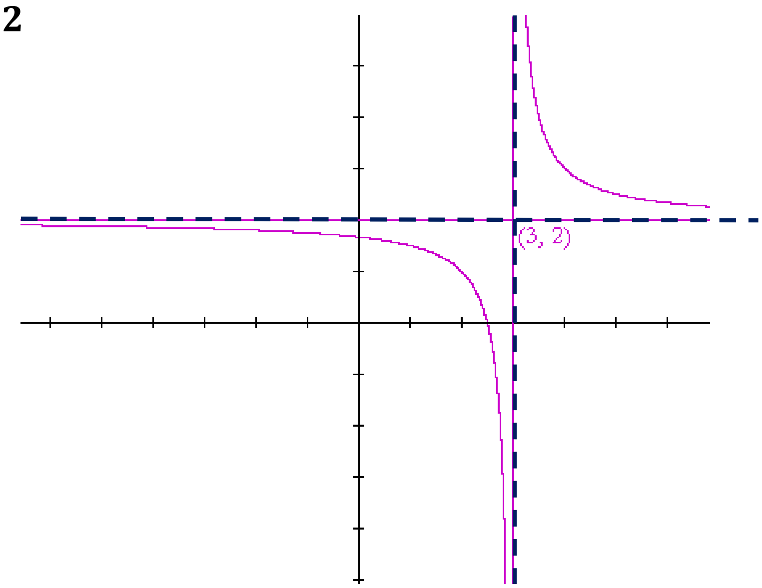
c) $y = \frac{-1}{x+2}$ **This is a reflection about the *x*-axis of graph b).**



What is the HA and VA?

d) $y = \frac{2x-5}{x-3} = \frac{1}{x-3} + 2$

What is the HA and VA?



Example 4.1.2: Page 306 vertical and horizontal asymptotes

Example 4.1.3: Page 307

Example 4.1.4: Page 309 List the horizontal asymptotes,

Example 5: Write the equation of the **vertical** and **horizontal asymptote**(s) of each of the following.

a) $y = \frac{3x+4}{x+1}$

c) $y = \frac{1}{x^2 - 16}$

b) $y = \frac{1}{2x+1}$

d) $f(x) = \frac{1}{x^2 + x - 6}$

Oblique or Slant Asymptotes

3) Definition (Oblique or Slant Asymptote (OA)**)**

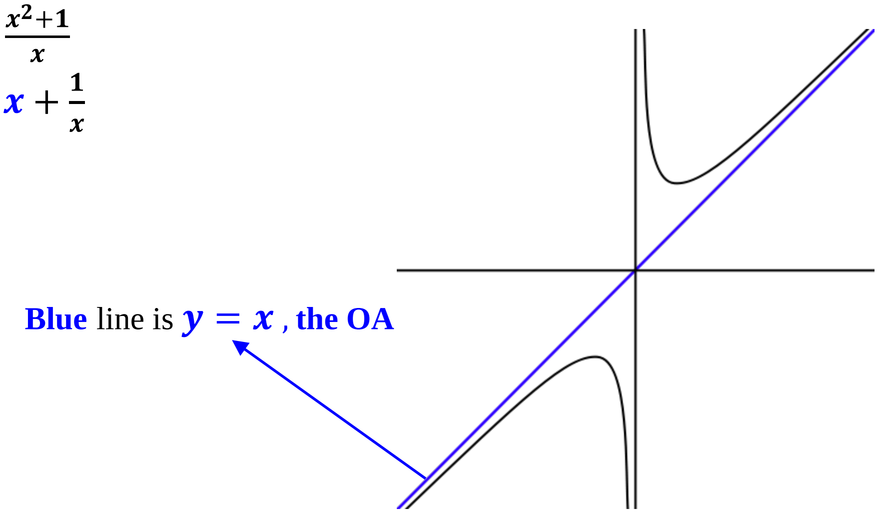
When a **linear asymptote** is **not parallel** to the ***x*- or *y*-axis**, it is called an **oblique asymptote** or **slant asymptote.**

Using Arrow Notations

The line $y = mx + b$ where $m \neq 0$ is called **a slant or oblique asymptote** of the graph of a function $y = f(x)$ if as $x \to \infty$ or as $x \to -\infty$, $f(x) \to mx + b$.

Example: Graph $y = \frac{x^2+1}{x}$

$y = x + \frac{1}{x}$



Question 4: What does the equation of an **oblique line** look like? **Ans.** $y = mx + b$

Example 4.1.6: Page 312 Find the slant asymptotes

Example 6: Find the Oblique Asymptote: (Use long division to find the OA)

a) $f(x) = \frac{2x^2 - 3x - 1}{x - 2}$

b) $f(x) = \frac{5x^3}{x^2 - 9}$

Asymptotes Summary

Vertical Asymptote

First simplify the rational expression; then if a is a zero of the new denominator, then the line $x = a$ is a vertical asymptote for the graph or the rational function.

Example 1: ***Find the vertical asymptotes:***

a) $f(x) = \dfrac{2}{x-4}$

b) $f(x) = \dfrac{5x}{x^2-9}$

c) $f(x) = \dfrac{x^2-16}{x-4} = \dfrac{(x-4)(x+4)}{(x-4)} = x+4,\ x \neq 4$ **No vertical asymptote**

Horizontal Asymptote

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \ldots + a_0}{b_k x^k + \ldots + b_0}$, **n =** degree of numerator, **k =** degree of denominator.

Then:

Condition on degrees	Asymptote
$n < k$	$y = 0$ line is H.A.
$n = k$	$y = \dfrac{a_n}{b_k}$ line is the H.A

Example 2: **Find the HA**

a) $f(x) = \dfrac{2}{x-4}$

b) $f(x) = \dfrac{2x^3+3x-7}{3x^3-5x^2+3x}$

Oblique (or Slant) Asymptote

Let $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + \ldots + a_0}{b_k x^k + \ldots + b_0}$, **n =** degree of numerator, **k =** degree of denominator.

Then:

Condition on degrees	Asymptote
$n > k$ by exactly 1	$y = Q(x)$, the quotient poly is O.A.
$n > k$ by more than 1	no H.A. or O.A.

Note: Graphs **can cross horizontal or oblique asymptotes, but** they **cannot cross vertical asymptotes!**

Example 3: Find the horizontal and/or oblique asymptotes:

a) $f(x) = \frac{5x}{x^2 - 9}$

$n = degree\ of\ numerator = 1$

$k = degree\ of\ denominator = 2$

$n < k$ ➔ **Line $y = 0$ is the horizontal asymptote**

b) $f(x) = \frac{5x^4 + 3x^2 + 2x - 8}{2x^2 + 2x - 8}$

$n = degree\ of\ numerator = 4$

$k = degree\ of\ denominator = 2$

$n > k + 1$ ➔ **No horizontal asymptote**

No oblique asymptote as well

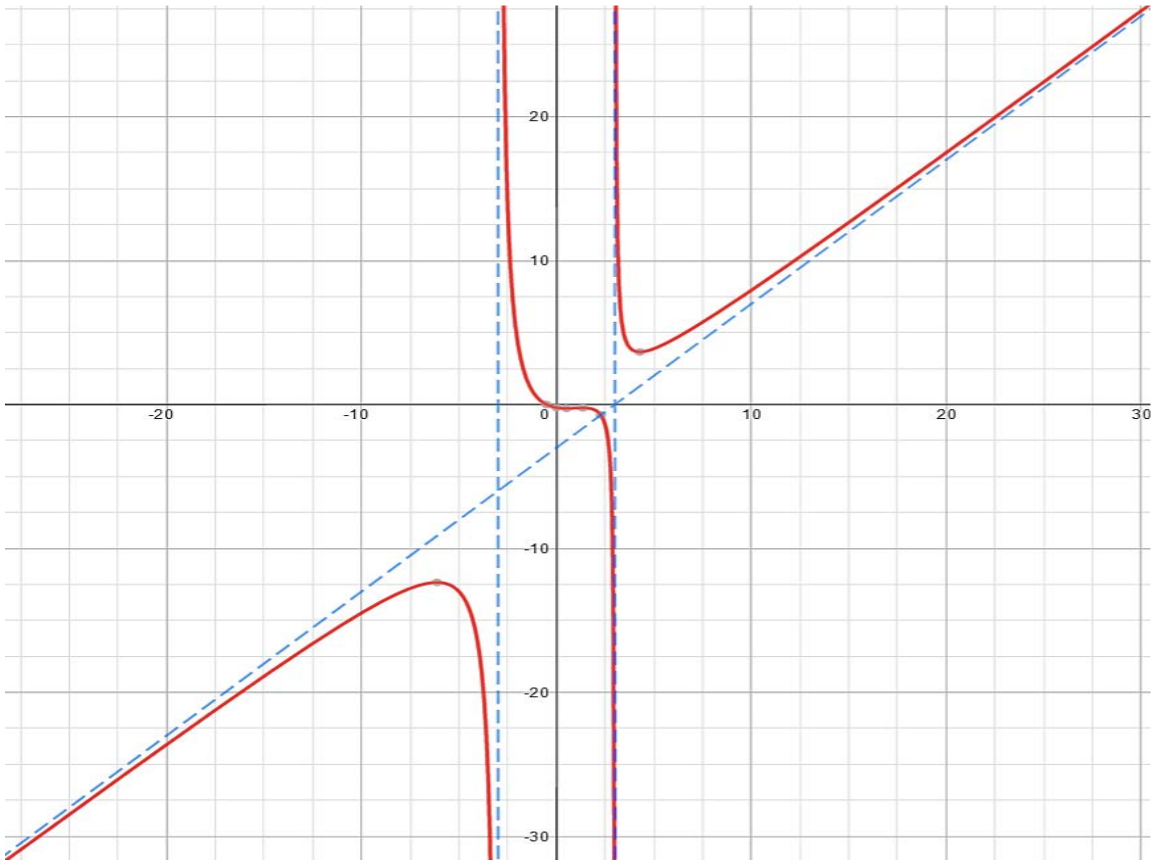
Example 4: For each of the following functions fill the table with the **correct asymptote(s) equation(s)**, otherwise write **none** if the function does not have the particular asymptote.

Functions	Vertical Asymptote(s)	Horizontal Asymptote	Oblique Asymptote
$f(x) = \frac{2}{x - 4}$	$x = 4$ line	$y = 0$ line	None
$f(x) = \frac{3x^2 + 1}{2x^2 - 4}$			
$f(x) = \frac{x^4 + 2x^3 - 6x^2 - 2}{x^3 + 1}$			
$f(x) = \frac{-x^2 + 3x}{2x^2 - 4x - 6}$			
$f(x) = \frac{3x^3 + 2x - 4}{x^2 - 4}$			

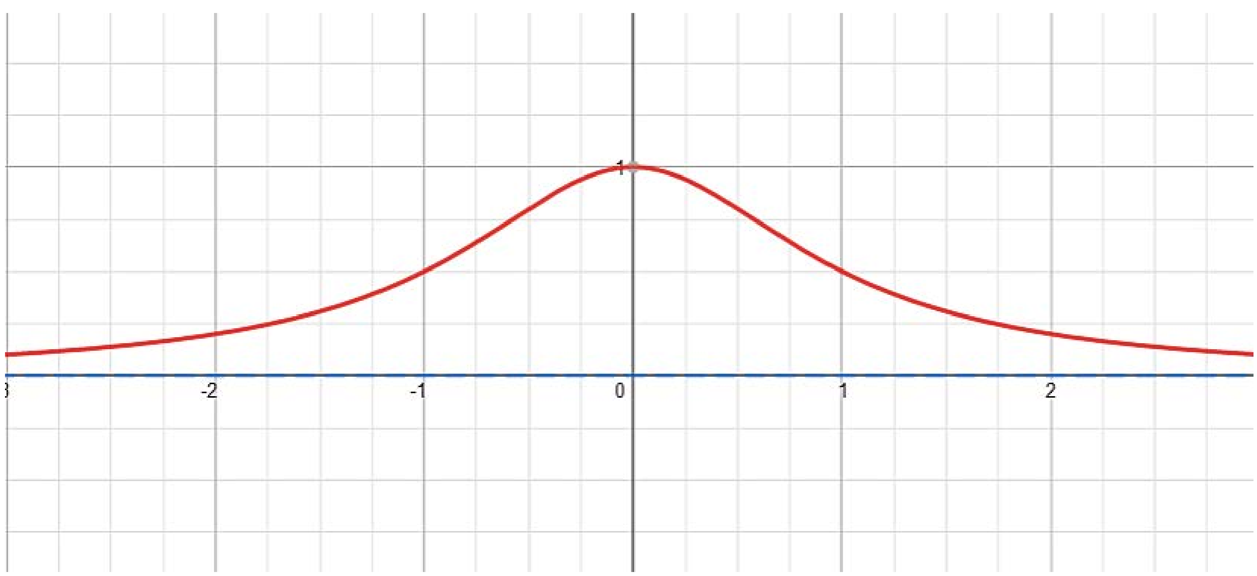
Graphs of Some Rational Functions

A) Find the domain and range and all asymptotes from the given graphs

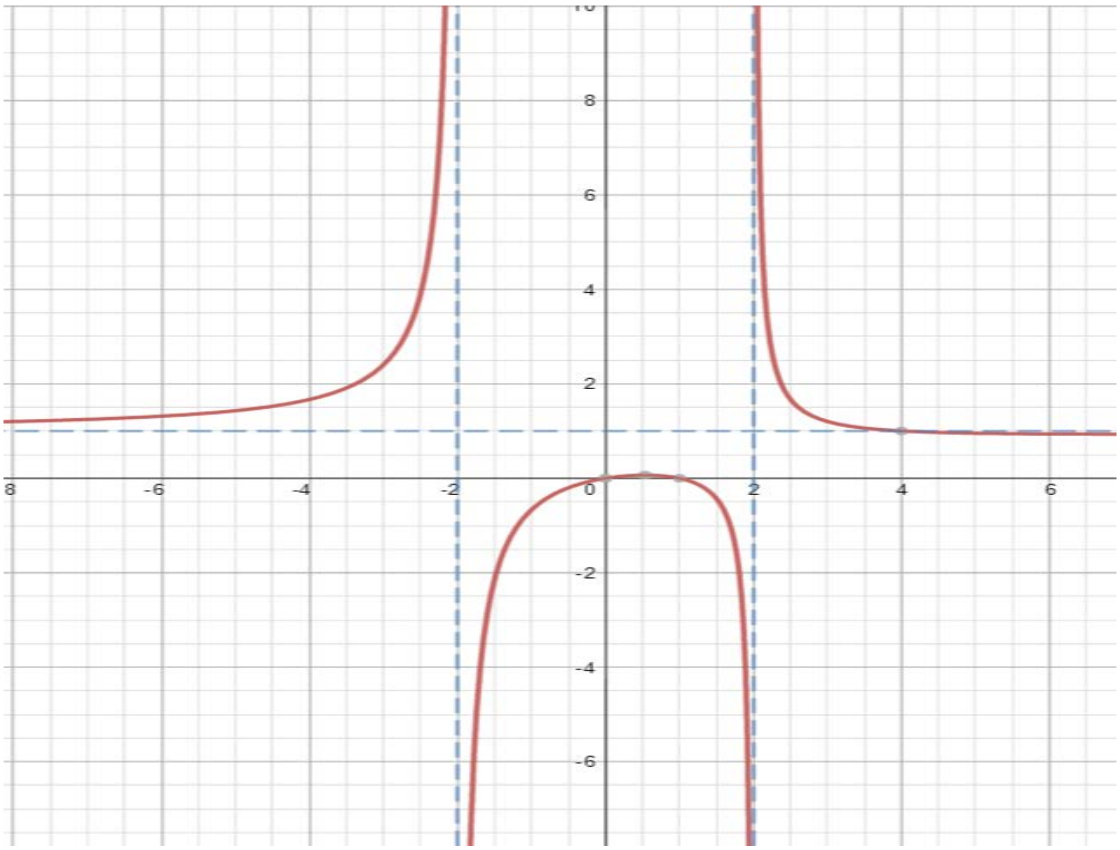
1) $y = \frac{x^3 - 3x^2 + 2x + 2}{x^2 - 9}$



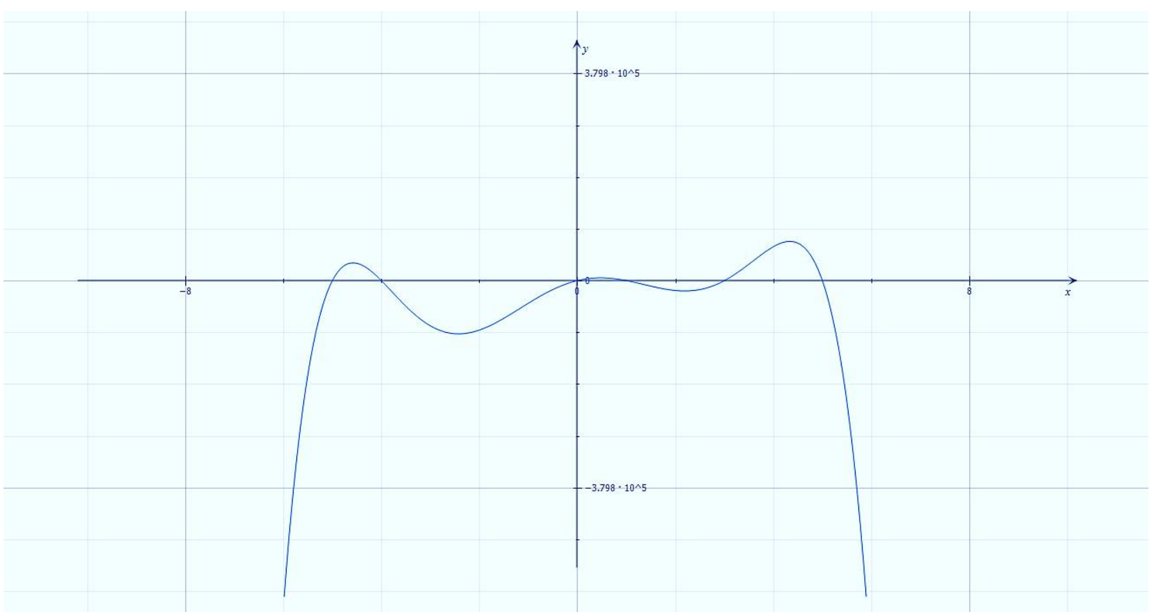
2) $y = \frac{1}{x^2 + 1}$



3) $y = \frac{3x^2 - x}{x^2 - 4}$



4) $y = (-2)x(x-1)(2x-10)(3x-9)(2x+8)(3x+15)$. This is a polynomial function as well; find the degree, leading coefficient, constant term and zeros.



B) Sketch the graphs of the following rational functions.

a) $f(x) = \frac{x}{x-2}$

b) $f(x) = \frac{3x^2+1}{2x^2-4}$

c) $f(x) = \frac{3x^3+2x-4}{x^2-4}$

d) $f(x) = \frac{-x^2+3x}{2x^2-4x-6}$

e) $f(x) = \frac{5x}{x^2-9}$

f) $f(x) = \frac{x^2-16}{x-4}$

Homework Practice Problems
Exercises 4.1.1: Page 314 #1 – 21 (odd numbers),

OER West Texas A&M Tutorial 40: Graphs of Rational Functions
Tutorial 41: Practice Test on Tutorials 34 - 40

Examples YouTube videos

- Asymptotes of rational functions: https://www.youtube.com/watch?v=2N62v_63SBo
- Finding vertical and horizontal asymptotes: https://www.youtube.com/watch?v=P0ZgqB44Do4
- Rational functions graphs 1: https://www.youtube.com/watch?v=ReEMqdZEEX0
- Graphs of rational functions 2: https://www.youtube.com/watch?v=p7ycTWq6BFk